Resolution enhancement of low resolution wavefields with POCS algorithm

A.E. Çetin, H. Özaktaş and H.M. Ozaktas

The problem of enhancing the resolution of wavefield or beam profile measurements obtained using low resolution sensors is addressed by solving the problem of interpolating signals from partial fractional Fourier transform information in several domains. The iterative interpolation algorithm employed is based on the method of projections onto convex sets (POCS).

Introduction: The fractional Fourier transform (FRT) has received significant interest since the early 1990s [1, 2]. The ath order FRT operation corresponds to the ath power of the ordinary Fourier transform operation. The zeroth-order FRT of a function is the function itself and the first-order FRT is equal to the ordinary Fourier transform. For additional properties and references, see [3].

In this Letter, an iterative algorithm for signal interpolation or extrapolation from partial FRT information is developed. More specifically, it is assumed that partial sets of samples are available for several a values. The problem is to interpolate or extrapolate the signal from this information. The reconstruction algorithm is globally convergent and it is based on the method of projections onto convex sets (POCS), a classical numerical technique [4–6]. The convergence of this algorithm can be proved easily in both continuous and discrete FRT domains because low resolution fractional Fourier measurements in both continuous and discrete domains correspond to closed and convex sets in L^2 or ℓ^2, respectively.

The FRT has been shown to describe the evolution of waves and beams as they propagate in space, in the Fresnel approximation [3]. Wavefields or beam profiles undergo continual fractional Fourier transformation as they propagate. Increasing values of a correspond to different measurement planes further along the direction of propagation. Therefore the problem solved here corresponds to the problem of reconstructing wavefields or beam profiles from partial measurements at more than one plane. This covers a wide range of applications where it is possible to make measurements in more than one plane, but not at every point or over the complete interval in any given plane. The availability of information from several planes is used to compensate the missing information in each of them.

Let us denote the ath order FRT operator as $\mathcal{F}^a$. The ath FRT $x_a = \mathcal{F}^a x$ of a function x is given by

$$x_a(u) = \sqrt{1 - i\cot\left(\frac{a\pi}{2}\right)} \int_{-\infty}^{\infty} \exp\left[i\pi\left(\cot\left(\frac{a\pi}{2}\right)u^2 - 2\csc\left(\frac{a\pi}{2}\right)uv + \cot\left(\frac{a\pi}{2}\right)v^2\right)\right]x(v)\,dv \qquad (1)$$

Positive and negative integer values of a simply correspond to repeated application of the ordinary forward and inverse Fourier transforms, respectively. The fractional Fourier transform operator satisfies index additivity: $\mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_2+a_1}$. $\mathcal{F}^4$ equals the identity operator.

The ath order discrete FRT $\mathbf{x}_a$ of an $N \times 1$ vector $\mathbf{x}$ is defined as $\mathbf{x}_a = \mathbf{F}^a\mathbf{x}$, where $\mathbf{F}^a$ is the $N \times N$ discrete FRT matrix [7], which is essentially the ath power of the ordinary discrete Fourier transform matrix $\mathbf{F}$. Let the discrete-time vector $\mathbf{x}$ contain the samples of the continuous time or space signal $x(v)$. If N is chosen equal to or greater than the time- or space-bandwidth product of the signal $x(v)$, then the discrete FRT approximates the continuous FRT.

In the problem considered, it is assumed that low resolution fractional Fourier domain measurements of $\mathbf{x}$ are available at several fractional domains, i.e. $x_{a_1}(kJ\Delta)$, $k = 0, \pm 1, \pm 2, \ldots, \pm K_1$, where J is an integer ≥ 2; and $x_{a_2}(kJ\Delta)$, $k = 0, \pm 1, \pm 2, \ldots, \pm K_2$, etc. are available for $a_1, a_2, \ldots, a_M$. The signal interpolation problem is the estimation of original signal samples $x_o(n\Delta)$, $n = 0, \pm 1, \pm 2, \ldots, \pm K_o$, from the above measurements.

Iterative signal recovery algorithm: The key idea of the POCS-based signal recovery algorithm is to obtain a solution which is consistent with all the available information [4–6]. In this method the set of all possible signals is assumed to constitute a Hilbert space with an associated norm in which the prior information about the desired signal can be represented in terms of convex sets. Let us suppose that the information about the desired signal is represented by M sets, C_m, $m = 1, 2, \ldots, M$. The desired signal must be in the intersection set $C_0 = \cap_{m=1}^{M} C_m$. Any member of the set C_0 is called a feasible solution [5]. If all of the sets C_m are closed and convex then a feasible solution can be found by making successive orthogonal projections onto the sets C_m. Let P_m be the orthogonal projection operator onto the set C_m. The iterates defined by the equation: $\mathbf{y}_{k+1} = P_1P_2, \ldots, P_M\mathbf{y}_k$, $k = 1, 2, \ldots$ converge to a member of the set C_0, regardless of the initial signal $\mathbf{y}_0$. In some problems, the set C_0 contains only the desired signal x_o. In this case, the iterates converge to x_o.

The interpolation problem can be posed in discrete spaces. Partial information in the discrete fractional Fourier domains can be represented as convex sets in ℓ^2. We define C_1 and C_2 in ℓ^2 as the set of signals the discrete FRTs of which are equal to $x_{a_1}(u_k)$ in $u_k \in U = \{u_k = kJ\Delta,\ k = 0, \pm 1, \pm 2, \ldots, K_1\}$ in the a_1st fractional domain, and $x_{a_2}(v_k)$ in $v_k \in V = \{v_k = kJ\Delta,\ k = 0, \pm 1, \pm 2, \ldots, K_2\}$ in the a_2nd fractional domain. The sets C_1 and C_2 are convex because the integral operator in (1) is a linear operator, or equivalently, the discrete FRT operator is a matrix. If data is also available in further fractional domains $a_3, a_4, \ldots$, then corresponding sets can be defined in a similar manner. If the original space-domain signal samples $x_o(n\Delta)$ are available for certain integer n values then this information can be modelled as a convex set in a similar manner as well, as the space domain is merely a special fractional domain with $a = 0$. Similarly, if the signal is known to be of finite extent, then this information can be modelled as a closed and convex set. Other space-domain information about the original signal including non-negativity and finite energy information belongs to the above class of sets. When such an information is available, it can be beneficially incorporated in our algorithm in a convenient manner to have robustness against noise.

Projection operations onto the sets $C_1, C_2, \ldots, C_M$ are straightforward to implement. Let $\mathbf{x}^{(l)}$ be the lth iterate of the iterative recovery process. Let $\mathbf{x}_a^{(l)}$ be the FRT of $\mathbf{x}^{(l)}$ in the ath domain. The projection operator replaces the FRT values of $x_a^{(l)}(u)$ in U

$$x_a^{(l+1)}(u_k) = x_a(u_k), \quad u_k \in U \qquad (2)$$

and retains the rest of the data outside the band U:

$$x_a^{(l+1)}(u_k) = x_a^{(l)}(u_k) \quad u \notin U \qquad (3)$$

The algorithm starts with an arbitrary initial estimate $\mathbf{y}_0 \in \mathrm{Ł}^2$. The initial estimate $\mathbf{y}_0$ is successively projected onto the sets C_m, $m = 1, 2, \ldots, M$, representing the partial fractional Fourier domain information in fractional domains a_m, $m = 1, 2, \ldots, M$, $0 \leq a_m \leq 1$ by using (2) and (3). In this manner the first iteration cycle is completed. This iterative procedure is repeated until a satisfactory level of error difference in successive iterations is obtained.

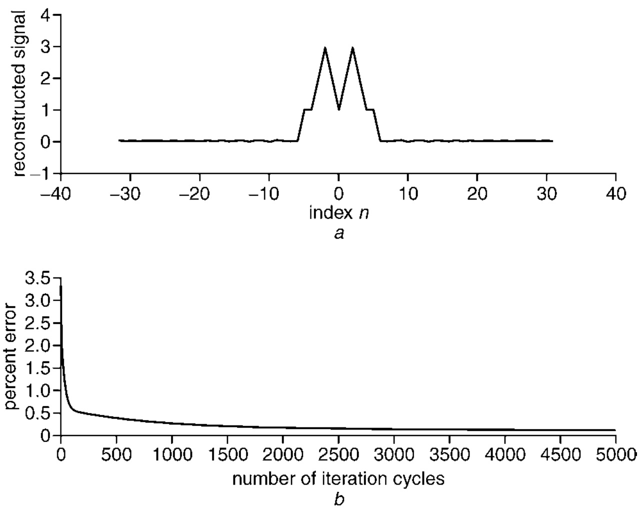

Fig. 1 *Reconstructed signal in first example, and per cent error against number of iteration cycles*

a Reconstructed signal
b Per cent error against number of iteration cycles

Simulation examples and conclusions: It is assumed that $a_1 = 0.5$th and $a_2 = 0.75$th order fractional Fourier domain measurements of the signal $\mathbf{x}_o = \{1, 1, 2, 3, 2, x_o(0) = 1, 2, 3, 2, 1, 1, 0, 0, \ldots\}$ are available on a uniform grid: $\mathbf{x}_{0.5}(k)$, $k = 1, 3, 5, \ldots, 63$, and $\mathbf{x}_{0.75}(k)$, $k = 2, 4, 6, \ldots, 64$ (the discrete FRT size is $N = 64$). That is, we know only every other sample in the discrete FRT domains of $a_1 = 0.5$ and $a_2 = 0.75$ (the sampling period is assumed to be $\Delta = 1$ without loss of generality). The original signal is almost perfectly reconstructed from the above information after 5000 iteration cycles. This corresponds to situations where the measuring device being used has only half the desired resolution and we are attempting to compensate for this by measuring the waves or beams at more than one plane.

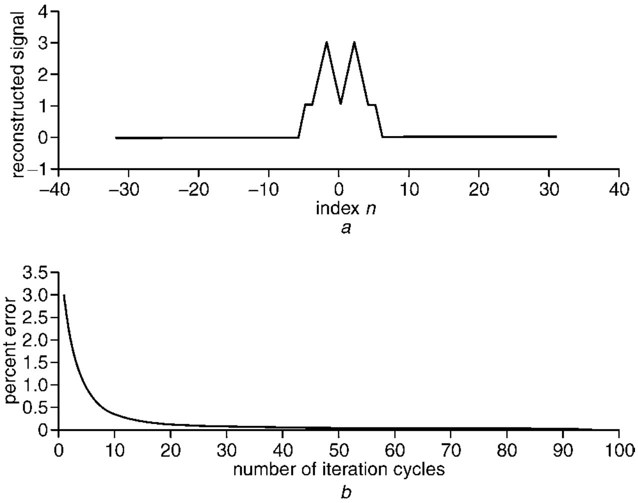

Fig. 2 *Reconstructed signal using finite support information, and per cent error against number of iteration cycles*

a Reconstructed signal
b Per cent error against number of iteration cycles

Let us consider another example in which partial measurements of the above original signal are available at $a_1 = 0.5$ and $a_2 = 0.75$, and $a_3 = 0.0$ which corresponds to the space (or time) domain: $\mathbf{x}_{0.5}(k)$, $k = 1, 3, 5, \ldots, 13$, and $\mathbf{x}_{0.75}(k)$, $k = 1, 3, 5, \ldots, 13$; and in $a_3 = 0$ space domain it is assumed that only even indexed samples of $\mathbf{x}_o(n)$ are available. The initial estimate is taken as $y_0(n) = 0$ for all n. The reconstructed signal obtained after 5000 iteration cycles is shown in Fig. 1*a*. The interpolated sample values are $x(\pm 1) = 2.004$, $x(\pm 3) = 1.986$, $x(\pm 5) = 1.001$. The per cent error against the number of iteration cycles is shown in Fig. 1*b*. The per cent restoration error is defined as follows: $100 \times \|\mathbf{y}_k - \mathbf{x}_o\|^2 / \|\mathbf{x}_o\|^2$ where $\mathbf{y}_k$ is the kth iterate.

If, in addition to the above information, it is assumed that the original signal has a finite support in the time domain, i.e. $x_0(n) = 0$, for $|n| \geq 10$, this constraint improves the speed of convergence as shown in Fig 2. Iterates converge to the desired solution after 100 iteration cycles as shown in Fig. 2*b*.

In general, if the number of observations is larger than or equal to the number of original signal samples to be estimated then iterates converge to a unique solution in noise-free cases. The performance of the algorithm under noisy observations is similar to the classical signal reconstruction problem involving only ordinary Fourier data. If data is available only in a narrow interval, then the reconstruction process can be noise sensitive as in the case of classical reconstruction from partial Fourier domain reconstruction problem. The existence of redundant data from several domains and the use of a finite energy set increases the noise robustness of the method.

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